The local recognition of reflection graphs of spherical Coxeter groups

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1 Introduction

Given a graph Γ one may ask to which extend it is determined by its local graphs, that is, by the induced subgraphs on the vertices adjacent to a particular vertex. If all these local graphs are isomorphic to a single graph Λ then Γ is said to be locally Λ. A classification of the graphs which are locally a given graph Λ is called a local recognition result. The local recognition of graphs has been studied extensively in the literature, for instance in [BH77], [HS85]. A particularly guiding example for the topic of the present article is the local recognition of the Kneser graphs studied in [Hal80] and [Hal87].

In this paper we are interested in the local recognition of Weyl graphs $\mathbb{W}(M)$ which are commuting graphs on the reflections of Coxeter groups. Summarizingly, we provide the following recognition results.

Theorem 1.1. *Let Γ be a connected graph.*

- *If $n \geqslant 6$ and Γ is locally $\mathbb{W}(A_n)$ then $\Gamma \cong \mathbb{W}(A_{n+2})$.*
- *If $n \geqslant 7$ and Γ is locally $\mathbb{W}(A_1 \sqcup D_n)$ then $\Gamma \cong \mathbb{W}(D_{n+2})$.*
- *If $n \geqslant 4$ and Γ is bichromatic locally homogeneous with $\Delta_s(\Gamma) \cong \mathbb{W}(B_{n+1})$ and $\Delta_\ell(\Gamma) \cong \mathbb{W}(A_1^\ell \sqcup B_n)$ then $\Gamma \cong \mathbb{W}(B_{n+2})$.*
- *If $n \geqslant 4$ and Γ is bichromatic locally homogeneous with $\Delta_s(\Gamma) \cong \mathbb{W}(A_1^s \sqcup C_n)$ and $\Delta_\ell(\Gamma) \cong \mathbb{W}(C_{n+1})$ then $\Gamma \cong \mathbb{W}(C_{n+2})$.*
- *If Γ is locally $\mathbb{W}(A_5)$ and $|\Gamma| = 36$ then $\Gamma \cong \mathbb{W}(E_6)$.*
- *If Γ is locally $\mathbb{W}(D_6)$ then $\Gamma \cong \mathbb{W}(E_7)$.*
- *If Γ is locally $\mathbb{W}(E_7)$ and $|\Gamma| = 120$ then $\Gamma \cong \mathbb{W}(E_8)$.*
- *If Γ is bichromatic locally homogeneous with $\Delta_s(\Gamma) \cong \mathbb{W}(B_3)$ and $\Delta_\ell(\Gamma) \cong \mathbb{W}(C_3)$, and $|\Gamma| = 24$ (or one of the conditions of Theorem 5.12) then $\Gamma \cong \mathbb{W}(F_4)$ or Γ is isomorphic to a second graph constructed during the proof of Proposition 5.13.*

The local recognition of the graphs $\mathbb{W}(A_n)$ and $\mathbb{W}(D_n)$ reduces to the recognition of the Kneser graphs which was first established in [Hal87]. The graphs $\mathbb{W}(E_n)$ are locally cotriangular graphs as well, and have as such been studied and locally recognized in [HS85]. The local recognition of $\mathbb{W}(B_n)$ and $\mathbb{W}(C_n)$ is proved in Theorem 4.4. Since the Weyl graph $\mathbb{W}(F_4)$ is not locally recognizable, compare Corollary 5.6, we study bichromatic graphs which are locally like $\mathbb{W}(F_4)$ and in the sequel obtain several characterizations of $\mathbb{W}(F_4)$, summarized in Theorem 5.12, as one of two tightest bichromatic graphs which are locally like $\mathbb{W}(F_4)$.

In the last section we turn to group theoretical applications of local recognition results for Weyl graphs. The paradigmatic and guiding result, exemplified in [GLS94, Theorem 27.1], is the characterization of the symmetric groups by means of the structure of its transposition centralizers. We give the following similar characterization, proved in Theorem 6.6, for the Coxeter group of type F_4 in terms of its reflection centralizers.

Theorem 1.2. *Let G be a group with nonconjugate involutions x, y such that*

- $C_G(x) = \langle x\rangle \times J$ *with* $J \cong W(B_3)$,
- $C_G(y) = \langle y\rangle \times K$ *with* $K \cong W(C_3)$,
- *x (respectively y) is a short (respectively long) root reflection in K (respectively J),*
- *$J \cap K$ contains involutions x_1, y_1 such that x_1 (respectively y_1) is a short (respectively long) root reflection in K as well as in J, and*
- *there are a long root reflection $y_0 \neq y, y_1$ in J and a short root reflection $x_0 \neq x, x_1$ in K such that x_0 and y_0 commute.*

If $G = \langle J, K\rangle$ then $G \cong W(F_4)$.

The interest in such group theoretic local recognition results stems from the classification of finite simple groups, outlined in [GLS94], and the fact that the majority of finite simple groups arises from (possibly twisted) Chevalley groups. These can be defined, due to the Curtis-Tits theorem in a version described in [Gra08, 4.1.3], similar to Coxeter groups as groups generated by SL(2, q) subgroups subject to certain relations, see [Pha70], [Hum72], [Tim04] and [Dun05]. Local recognition results for instance for Chevalley groups of type A_n, $n \geqslant 8$, based on graph theoretical results have been studied in [Gra02], [Gra04]. Recently, see [Gra08], the first author and Kristina Altmann proved a local recognition result for Chevalley groups of type A_7 and E_6 based on results and techniques of [Alt07] and making use of the local recognition of graphs that are locally $\mathbb{W}(A_5)$. We hope that our analysis can help to approach a similar recognition result for Chevalley groups of type F_4. For more details we refer to the thesis [Str08] of the third author.

In section 2 we start with discussing the local recognition of graphs in general and review some known recognition results. Reflection graphs of Coxeter groups are introduced in section 3 and locally recognized in section 4 while the case F_4 is seperately studied in section 5. Finally, we close with group theoretic applications in section 6.

2 Local recognition of graphs

All graphs considered in this text are simple and undirected. We use $\perp$ to denote adjacency, and our notation for operations on graphs like the cartesian product or joins follows [Har94]. Let Γ be a graph. We write $x^\perp$ to denote the set of neighbors of x, that is, the set of vertices adjacent to x. The induced subgraph on $x^\perp$ is called the *local graph* at x, and Γ is said to be *locally homogeneous* if all its local graphs are isomorphic to some graph Δ. In this case, Γ is said to be locally Δ, and Δ is referred to as the local graph of Γ. If Γ is locally homogeneous then we denote its local graph by $\Delta(\Gamma)$.

Remark 2.1. In literature, some authors refer to the local graph at a vertex x as the *link* of x. Accordingly, locally homogeneous graphs are also referred to as graphs with constant link. This terminology stems from the fact that when considering a graph as a simplicial complex the notion of the local graph at a vertex coincides with the already existing notion of the link of a vertex.

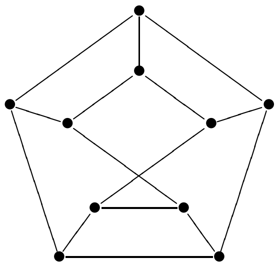
Remark 2.2. Which graphs occur as local graphs? This question was posed by Alexander A. Zykov in [Zyk64] and is commonly, compare [Bug90], referred to as the *Trahtenbrot-Zykov problem*. An introduction to this problem can be found for instance in the first part of [BC75]. A comprehensive resource for techniques for proving whether a graph is a local graph is [Hal85]. Moreover, [Hal85] describes the graphs of order up to 6 that are local graphs.

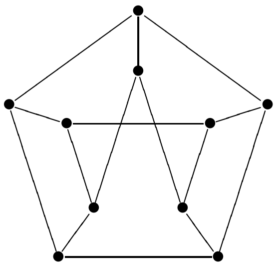
If Γ_1 is locally Δ_1, and Γ_2 is locally Δ_2 then the cartesian product $\Gamma_1 \times \Gamma_2$ is locally the disjoint union $\Delta_1 \sqcup \Delta_2$. This easy observation shows that the class of all local graphs is closed under disjoint unions. In [Bul73] it is shown that there is no algorithm to decide which graphs are local graphs when infinite graphs are allowed. When restricting to finite graphs this question is still open as is noted in [Bug90]. To illustrate that it makes a difference to permit infinite graphs we refer to [Hal85, 4.16] where it is proved that there is no finite graph Γ that is locally $K_{1,3} \sqcup \overline{K_2}$ while infinite graphs are constructed which are locally $K_{1,3} \sqcup \overline{K_2}$.

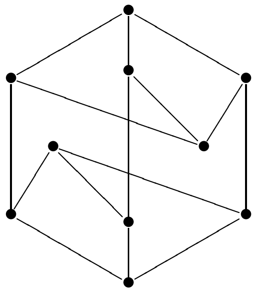
Here, we will be interested in the problem of characterizing a locally homogeneous graph in terms of its local graph. Notice that a graph Γ is locally Δ if and only if all its connected components are locally Δ. Accordingly, we usually restrict the discussion to connected graphs, and we say that a connected locally homogeneous graph Γ is *locally recognizable* if up to isomorphism Γ is the only connected graph that is locally $\Delta(\Gamma)$. In case Λ is another locally homogeneous graph such that $\Delta(\Lambda) \cong \Delta(\Gamma)$ we say that Λ is *locally like* Γ.

Example 2.3. One easily verifies that a connected graph Γ is locally K_{n-1} if and only if $\Gamma \cong K_n$. Hence the complete graphs K_n are locally recognizable. The circuit graphs C_n, on the other hand, are not locally recognizable as they all are locally $\overline{K_2}$ provided that $n \geqslant 4$.

Remark 2.4. If a graph Γ is transitive then Γ is locally homogeneous. Not surprisingly, the converse is not true. The smallest graphs which are locally homogeneous but not transitive have 10 vertices, and up to isomorphism there are exactly three such graphs, depicted below.







These graphs and further information can be found in [Hal85].

One easily verifies that the Kneser graph $K(n,\, k)$ is locally homogeneous with local graph $K(n-k,\, k)$. The second author proves in [Hal87] that for n sufficiently large compared to k the Kneser graphs are locally recognizable, and classifies in [Hal80] the three connected graphs which are locally the Petersen graph $K(5,\, 2)$. Finally, the classification of graphs that are locally $K(6,2)$ is contained in [HS85].

Theorem 2.5. ([Hal87], [Hal80], [HS85]) *Let $k \geqslant 1$, and Γ be a connected graph that is locally $K(n,k)$.*

- *If $n \geqslant 3k+1$ then $\Gamma \cong K(n+k,k)$.*
- *If $(n, k) = (5, 2)$ then Γ is isomorphic to one of the graphs $K(7, 2)$, $3 \cdot K(7, 2)$, or $\Sigma L_{2,25}$. In particular, $|\Gamma| \in \{21, 63, 65\}$.*
- *If $(n,k) = (6,2)$ then Γ is isomorphic to one of the graphs $K(8,2)$, $\mathcal{S}p_2(6)$ minus $\{x\} \cup x^{\perp}$ for some x, or $\mathcal{NS}p^{-}(6)$. In particular, $|\Gamma| \in \{28, 32, 36\}$.* □

Here, the graph $3 \cdot K(7,2)$ is the 3-fold cover of $K(7,2)$, and $\Sigma L_{2,25}$ is the graph on the conjugates of the unique nontrivial field automorphism of $\mathbb{F}_{25}$ in the special semilinear group $\Sigma L(2, 25)$ with two elements adjacent whenever they commute. More details can be found in [Hal80]. Further, the graph $\mathcal{S}p_2(2n)$ is the graph on the nonzero vectors of $V = \mathbb{F}_2^{2n}$ with two vectors adjacent whenever they are perpendicular with respect to a nondegenerate symplectic form B on V. Up to isomorphism there are only two quadratic forms Q^{+} and Q^{-}, corresponding to maximal or minimal Witt index, on V that B is associated to, and the graph $\mathcal{NS}p^{\varepsilon}(2n)$ is the induced subgraph of $\mathcal{S}p_2(2n)$ on the vectors that are nonsingular under Q^{ε}. For more details about these graphs and symplectic forms over V we refer to [HS85].

The second author and Ernest E. Shult actually prove a lot more in [HS85]. They characterize the graphs that are locally cotriangular in the following sense. A graph is said to be *cotriangular* if every pair x, y of nonadjacent vertices is contained in a cotriangle, that is, a 3-coclique $\{x, y, z\}$ such that every other vertex is adjacent to either all or exactly one of the vertices x, y, z. Observe that a join $\Gamma + \Lambda$ is cotriangular if and only if both Γ and Λ are. Denote with Γ^* the *reduced graph* of Γ, that is the graph on the equivalence classes of vertices of Γ with the same closed neighborhood and two classes adjacent whenever some representatives are adjacent. Then Γ is cotriangular if and only if Γ^* is. A graph Γ is called completely reduced in this context whenever $\Gamma^* = \Gamma$ and Γ can not be decomposed into $\Gamma_1 + \Gamma_2$ with nonempty Γ_1, Γ_2. A classification of all cotriangular graphs is given by the following theorem.

Theorem 2.6. ([HS85, Cotriangle Theorem]) *A finite completely reduced graph is cotriangular if and only if it is isomorphic to one of the graphs*

$$K(n,2),\ n \geqslant 2; \quad \mathcal{S}p_2(2n),\ n \geqslant 2; \quad \mathcal{NS}p^{\varepsilon}(2n),\ \varepsilon = \pm 1,\ n \geqslant 3. \qquad \square$$

The graphs $K(2,2) \cong K_1$ and $K(3,2) \cong \overline{K_3}$ are considered degenerate. Let $\mathcal{D}$ denote the set of graphs Γ such that Γ^* is a finite completely reduced cotriangular graph. If $\mathcal{G}$ is a collection of graphs then we say that a graph Γ is locally $\mathcal{G}$ if for each $x \in \Gamma$ the local graph at x is isomorphic to some graph of $\mathcal{G}$.

Theorem 2.7. ([HS85, Main Theorem]) *Let Γ be connected and locally $\mathcal{D}$. Then either Γ is locally $\{K_1, \overline{K_3}\}$ or Γ is isomorphic to one of the following graphs*

- *$K(n,2)$ where $n \geqslant 7$,*
- *$\mathcal{S}p_2(2n)$ possibly with a polar subspace deleted,*
- *$\mathcal{H}_{2n}^{\varepsilon}(T)$, $\mathcal{G}_{2n}^{\varepsilon}$,*
- *$3 \cdot K(7,2)$, $\Sigma L_{2,25}$, or $\mathcal{N}_6^{+}(3)$.* □

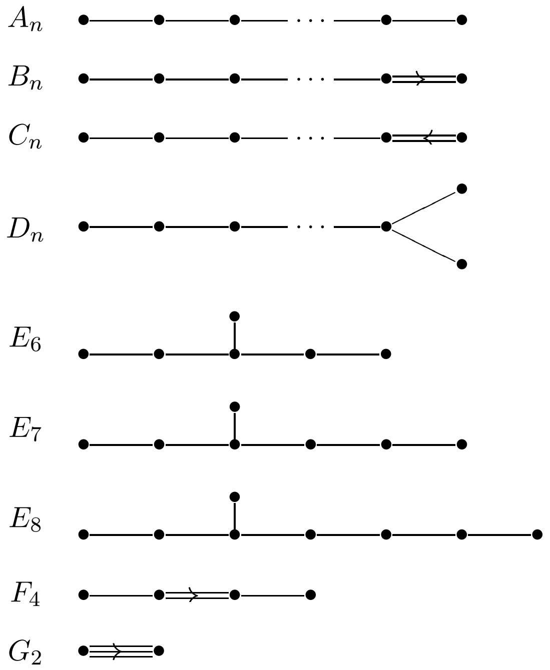
The graphs $\mathcal{H}^{\varepsilon}_{2n}(T)$, $\mathcal{G}^{\varepsilon}_{2n}$ are derived from the graph $\mathcal{S}p_2(2n)$. For a precise definition and description of these graphs as well as $\mathcal{N}^{+}_{6}(3)$ we refer to [HS85]. Note that the case $k = 2$ of Theorem 2.5 as well as Theorem 2.5 and Theorem 2.5 can be regarded as special cases of the classification in Theorem 2.7. Likewise, the following result which can also be found in [BH77] can be obtained from Theorem 2.7.

Theorem 2.8. ([HS85, Theorem 5]) *Let Γ be connected and locally $\mathcal{S}p_2(2n)$ for some $n \geqslant 2$. Then Γ is isomorphic to one of the following graphs $\mathcal{N}\mathcal{S}p^{+}(2n+2)$, $\mathcal{N}\mathcal{S}p^{-}(2n+2)$, or $\mathcal{S}p_2(2n+2)$ minus $\{x\} \cup x^{\perp}$ for some x.* □

3 Reflection graphs on Coxeter groups

We assume that the reader is familiar with Coxeter groups and root systems. A warmly recommended introduction is [Hum92] or the classical [Bou02]. The *commuting graph* of a group G on $X \subseteq G$ is the graph with vertices X in which two vertices $g, h \in X$ are adjacent whenever g and h commute. We will study the commuting graphs of finite Coxeter groups on their reflections. Since we are interested in local recognition results we will focus on finite irreducible Coxeter groups for which the reflection graph is locally homogeneous. The cases H_3, H_4 and $I_2(m)$ are not interesting for the purpose of local recognition, compare [Str08], so we may further restrict to Coxeter groups which arise from crystallographic root systems. These are classified in the following well-known theorem, see for instance [Hum92] or [Bou02].

Theorem 3.1. *An irreducible root system is crystallographic if and only if its Dynkin diagram is isomorphic to one of the diagrams A_n for $n \geqslant 1$, B_n for $n \geqslant 2$, C_n for $n \geqslant 2$, D_n for $n \geqslant 4$, E_6, E_7, E_8, F_4, or G_2 which are depicted below (the subindex corresponding to the number of vertices, and arrows pointing towards short roots).*



□

Let Φ be a crystallographic root system with Dynkin diagram M. Note that upon decomposing Φ into irreducible root systems as classified in Theorem 3.1 each root of Φ can be considered either short or long (with the agreement that in the absense of two distinct root length every root is long). We denote with $W(M)$ the Weyl group of Φ, i.e. the group generated by the reflections through the roots of Φ, together with the notion of a short (respectively long) root reflection by $W(M)$. The *Weyl graph* $\mathbb{W}(M)$ is the commuting graph of $W(M)$ on its reflections. Assume that M is connected. If M is simply laced then all reflections in $W(M)$ are conjugate, see for instance [Hum92] or [Bou02]. The Weyl graph $\mathbb{W}(M)$ is therefore locally homogeneous. On the other hand, if M is not simply laced then there are two conjugacy classes of reflections in $W(M)$, namely short and long root reflections, and we regard $\mathbb{W}(M)$ as a bichromatic graph. Instead of assigning arbitrary colors we accordingly refer to the vertices of $\mathbb{W}(M)$ corresponding to short (respectively long) root reflections as short (respectively long) vertices. If we naturally extend the notion of being locally homogeneous to bichromatic graphs, see section 4, the Weyl graph $\mathbb{W}(M)$ is locally homogeneous again.

Remark 3.2. The (long) local graph of a Weyl graph $\mathbb{W}(M)$ can be obtained in a somewhat surprising way as follows. Given a connected Dynkin diagram M consider the corresponding affine diagram, which is just M together with the highest root added, and delete the highest root together with all its neighbors. Then the (long) local graph of $\mathbb{W}(M)$ is the Weyl graph corresponding to this diagram, cf. [Hum92, 2.11] and [Str08, 1.83]. A general way to find the centralizer subgroups of reflections in arbitrary Coxeter groups is established in [Bri96].

Notice that the reflections through roots α and β commute if and only if the roots α and β are orthogonal. Accordingly, the Weyl graph $\mathbb{W}(M)$ can be constructed as the graph on the roots of Φ, with roots α and $-\alpha$ identified for each $\alpha \in \Phi$, such that two roots are adjacent whenever they are orthogonal. In order to describe the graphs $\mathbb{W}(M)$, we therefore provide constructions for the irreducible crystallographic root systems classified in Theorem 3.1. These descriptions are based on [Hum92, 2.10] and [Bou02]. Throughout, let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ denote the standard basis of $\mathbb{R}^n$, and L_n the standard lattice $\mathbb{Z}\varepsilon_1 + \mathbb{Z}\varepsilon_2 + \ldots + \mathbb{Z}\varepsilon_n$.

Example 3.3. Let $\Phi(A_n)$ be the vectors of squared length 2 in the standard lattice L_{n+1} which are orthogonal to $\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_{n+1}$ so that $\Phi(A_n)$ consists of the $n(n+1)$ roots $\varepsilon_i - \varepsilon_j$, $1 \leqslant i \neq j \leqslant n+1$. The Weyl group $W(A_n)$ generated by the root system $\Phi(A_n)$ is the symmetric group Sym_{n+1} which faithfully acts on $\mathbb{R}^{n+1}$ by permuting the basis vectors $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n+1}$. For $i < j$ denote with $y_{i,j}$ the reflection through $\pm(\varepsilon_i - \varepsilon_j)$. $\mathbb{W}(A_n)$ is the graph with vertices $y_{i,j}$, $1 \leqslant i < j \leqslant n+1$, such that $y_{i,j} \perp y_{k,l}$ if and only if $\{i, j\} \cap \{k, l\} = \emptyset$. Consequently, the Weyl graph $\mathbb{W}(A_n)$ is isomorphic to the Kneser graph $K(n+1, 2)$. In particular, $\mathbb{W}(A_n)$ is connected if and only if $n \geqslant 4$.

Example 3.4. Let $n \geqslant 2$, and let $\Phi(B_n)$ be the vectors of squared length 1 or 2 in the standard lattice L_n. Accordingly, $\Phi(B_n)$ consists of the $2n$ short roots $\pm\varepsilon_i$, $1 \leqslant i \leqslant n$, and the $2n(n-1)$ long roots $\pm\varepsilon_i \pm \varepsilon_j$, $1 \leqslant i \neq j \leqslant n$. The Weyl group $W(B_n)$ generated by $\Phi(B_n)$ is the signed permutation group $(\mathbb{Z}/2)^n \rtimes \mathrm{Sym}_n$ which acts faithfully on $\mathbb{R}^n$ by changing the signs of and permuting the basis vectors $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$. Denote with $y_{i,i}$ the reflection through $\pm\varepsilon_i$. Further, for $i < j$, denote with $y_{i,j}$ the reflection through $\pm(\varepsilon_i - \varepsilon_j)$ and with $y_{j,i}$ the reflection through $\pm(\varepsilon_i + \varepsilon_j)$. Accordingly, we see that $\mathbb{W}(B_n)$ is the bichromatic graph with vertices $y_{i,j}$, $1 \leqslant i, j \leqslant n$, where the $y_{i,i}$ are short and the $y_{i,j}$ with $i \neq j$ are long vertices, and $y_{i,j} \perp y_{k,l}$ if and only if $\{i, j\} \cap \{k, l\} = \emptyset$ or $(k, l) = (j, i)$. In particular, $\mathbb{W}(B_n)$ is connected if and only if $n \geqslant 3$.

Example 3.5. $\Phi(C_n)$ is the dual root system of $\Phi(B_n)$ which is obtained by switching the role of short and long roots. $\Phi(C_n)$ consists of the $2n(n-1)$ short roots $\pm\varepsilon_i \pm \varepsilon_j$, $1 \leqslant i \neq j \leqslant n$, and the $2n$ long roots $\pm 2\varepsilon_i$, $1 \leqslant i \leqslant n$. Clearly, the Weyl group $W(C_n)$ as well as the Weyl graph $\mathbb{W}(C_n)$ is obtained from $W(B_n)$ respectively $\mathbb{W}(B_n)$ by exchanging short and long root reflections respectively vertices.

Example 3.6. Let $\Phi(D_n)$ be the vectors of squared length 2 in the standard lattice L_n. Consequently, $\Phi(B_n)$ consists of the $2n(n-1)$ roots $\pm\varepsilon_i \pm \varepsilon_j$, $1 \leqslant i \neq j \leqslant n$. Observe that $\Phi(D_n) \subset \Phi(B_n)$. For $i < j$, as in Example 3.4, denote with $y_{i,j}$ the reflection through $\pm(\varepsilon_i - \varepsilon_j)$, and with $y_{j,i}$ the reflection through $\pm(\varepsilon_i + \varepsilon_j)$. Accordingly, $\mathbb{W}(D_n)$ is the graph with vertices $y_{i,j}$, $1 \leqslant i, j \leqslant n$, such that $y_{i,j} \perp y_{k,l}$ if and only if $\{i,j\} \cap \{k,l\} = \emptyset$ or $(k,l) = (j,i)$. $\mathbb{W}(D_n)$ is therefore isomorphic to the composition graph $K(n,\ 2)[K_2]$, that is, the graph arising from the Kneser graph $K(n,2)$ by replacing each vertex by an adjacent pair of vertices. In particular, $\mathbb{W}(D_n)$ is connected if and only if $n \geqslant 5$. Further, the reduced graph $\mathbb{W}(D_n)^*$ is isomorphic to $K(n,2)$ provided that $n \geqslant 5$.

Example 3.7. We omit the descriptions of the root systems of type E_6, E_7 and E_8 for which we refer to [Hum92, 2.10], [Bou02] or [Str08]. It is well-known and, for example, verified in [Str08] that

- $\mathbb{W}(E_6) \cong \mathcal{NS}p^-(6)$,
- $\mathbb{W}(E_7) \cong \mathcal{S}p_2(6)$,
- $\mathbb{W}(E_8) \cong \mathcal{NS}p^+(8)$.

Example 3.8. Let $\Phi(F_4)$ consist of the vectors of squared length 1 or 2 in $L_n + \mathbb{Z}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)/2$. $\Phi(F_4)$ then contains the $8 + 16 = 24$ short roots $\pm\varepsilon_i$, $1 \leqslant i \leqslant 4$, and $(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)/2$ as well as the 24 long roots $\pm\varepsilon_i \pm \varepsilon_j$, $1 \leqslant i \neq j \leqslant 4$. We will discuss the Weyl graph $\mathbb{W}(F_4)$ in detail in section 5.

Example 3.9. Let $\Phi(G_2)$ be the vectors of squared length 2 or 6 in the standard lattice L_3 which are orthogonal to $\varepsilon_1 + \varepsilon_2 + \varepsilon_3$. Consequently, $\Phi(G_2)$ consists of the 6 short roots $\varepsilon_i - \varepsilon_j$, $1 \leqslant i \neq j \leqslant 3$, and the 6 long roots $\pm(2\varepsilon_i - \varepsilon_j - \varepsilon_k)$, $\{i,j,k\} = \{1,2,3\}$. Notice that each root of $\Phi(G_2)$ is orthogonal to exactly one other root, and that two such orthogonal roots are of different type. The Weyl graph $\mathbb{W}(G_2)$ is thus isomorphic to three disjoint edges of mixed type.

4 Local recognition of Weyl graphs

In the case that M is a simply laced connected Dynkin diagram the Weyl graphs $\mathbb{W}(M)$ are locally cotriangular graphs which have been studied and locally recognized in [HS85] as was presented in section 2.

Theorem 4.1. *Let $n \geqslant 6$. A connected graph Γ is locally $\mathbb{W}(A_n)$ if and only if $\Gamma \cong \mathbb{W}(A_{n+2})$.*

Proof. By Example 3.3, $\mathbb{W}(A_n)$ is isomorphic to the Kneser graph $K(n+1,2)$. Accordingly, Theorem 2.5 applies. □

Theorem 4.2. *Let $n \geqslant 7$. A connected graph Γ is locally $\mathbb{W}(A_1 \sqcup D_n)$ if and only if $\Gamma \cong \mathbb{W}(D_{n+2})$.*

Proof. Exploiting the assumed local structure, we see that the vertices of Γ come in pairs. Namely, for each vertex x there exists a unique vertex x' such that the closed neighborhoods of x and x' agree. Therefore $\Gamma \cong \Gamma^*[K_2]$. With Example 3.6 in mind, the reduced graph Γ^* is seen to be connected and locally $\mathbb{W}(A_{n-1})$. By Theorem 4.1, $\Gamma^* \cong \mathbb{W}(A_{n+1})$, and the claim follows. ☐

Theorem 4.3. *Let* Γ *be a connected graph.*

- *If* Γ *is locally* $\mathbb{W}(A_5)$ *and* $|\Gamma| = 36$ *then* $\Gamma \cong \mathbb{W}(E_6)$.
- *If* Γ *is locally* $\mathbb{W}(D_6)$ *then* $\Gamma \cong \mathbb{W}(E_7)$.
- *If* Γ *is locally* $\mathbb{W}(E_7)$ *and* $|\Gamma| = 120$ *then* $\Gamma \cong \mathbb{W}(E_8)$.

Proof. According to Theorem 2.5 there are three connected graphs that are locally $\mathbb{W}(A_5)$. Among these, $\mathcal{NS}p^-(6)$ is the only graph with 36 vertices. The claim therefore follows from the isomorphism $\mathbb{W}(E_6) \cong \mathcal{NS}p^-(6)$ stated in Example 3.7. For the second claim, recall from Example 3.6 that $\mathbb{W}(D_6)^* \cong K(6, 2)$. Hence Theorem 2.7 applies, and we can use $\mathbb{W}(E_7) \cong \mathcal{S}p_2(6)$. Likewise, since $\mathbb{W}(E_8) \cong \mathcal{NS}p^+(8)$ the third claim follows from Theorem 2.8. ☐

Let M be a connected Dynkin diagram, as classified in Theorem 3.1, which is not simply laced. Recall that this means that the root system $\Phi(M)$ contains roots of two lengths, so that, consequently, the Weyl graph $\mathbb{W}(M)$ is bichromatic. For a graph Γ we denote with Γ^s (respectively Γ^ℓ) the bichromatic graph obtained from Γ by considering all vertices as short (respectively long). In the sequel we are interested in local recognition results for the Weyl graphs $\mathbb{W}(B_n)$, $\mathbb{W}(C_n)$ and $\mathbb{W}(F_4)$. Extending our previous terminology we say that a bichromatic graph is *locally homogeneous* if the local graphs at short vertices are all isomorphic to some bichromatic graph Δ_s and if the local graphs at long vertices are all isomorphic to some bichromatic graph Δ_ℓ. In this case we say that Δ_s is the *short local graph* of Γ and that Δ_ℓ is the *long local graph* of Γ. If Γ is a bichromatic locally homogeneous graph then we denote its short local graph by $\Delta_s(\Gamma)$ and its long local graph by $\Delta_\ell(\Gamma)$. If Λ is another bichromatic locally homogeneous graph such that $\Delta_s(\Lambda) \cong \Delta_s(\Gamma)$ as well as $\Delta_\ell(\Lambda) \cong \Delta_\ell(\Gamma)$ then we say that Λ is *locally like* Γ. Finally, for any graph Γ and $X \subseteq \Gamma$ we set $X^\perp = \bigcap_{x \in X} x^\perp$.

Theorem 4.4. *Let* $n \geqslant 4$, *and let* Γ *be a connected bichromatic locally homogeneous graph with* $\Delta_s(\Gamma) \cong \mathbb{W}(B_{n+1})$ *and* $\Delta_\ell(\Gamma) \cong \mathbb{W}(A_1^\ell \sqcup B_n)$. *Then* $\Gamma \cong \mathbb{W}(B_{n+2})$.

Proof. Let X be a short component of Γ and $x \in X$ a short vertex. The short induced subgraph on $x^\perp$ is a clique on $n+1$ elements which implies that X is a clique on $n+2$ elements. By assumption, the long neighbors of x induce a subgraph isomorphic to the long induced subgraph of $\mathbb{W}(B_{n+1})$. This subgraph is isomorphic to $\mathbb{W}(D_{n+1})$ and, in particular, is connected for $n \geqslant 4$, see Example 3.6. This implies that all long neighbors of x are contained in a single long component Y of Γ. Consider a short vertex $x_1 \in X$ adjacent to x. Again, all long neighbors of x_1 lie in one long component of Γ. But looking at $\{x, x_1\}^\perp \subset x^\perp$ we see that x and x_1 share long neighbors whence this component has to be Y as well. Since X is connected this shows that all long vertices adjacent to some vertex of X are contained in Y. Likewise, let $y \in Y$. The short induced subgraph of $y^\perp$ is a clique on n vertices and thus in particular connected. Again, we see that for a long vertex y_1 adjacent to y the common neighbors $\{y, y_1\}^\perp$ contain a short vertex. Therefore the same argument as before shows that all short vertices adjacent to some vertex of Y are contained in X. Since Γ is connected this proves that X and Y are the only short respectively long components of Γ.

We count the number of long vertices by counting the long neighbors of the $n+2$ short vertices of Γ. By assumption, a short vertex has $(n+1)n$ long neighbors. Further, two short vertices have $n(n-1)$ long neighbors in common, three short vertices have $(n-1)(n-2)$ long neighbors in common, and so on. Thus there are

$$\binom{n+2}{1}(n+1)n-\binom{n+2}{2}n(n-1)+\ldots+(-1)^{n+1}\binom{n+2}{n}2=(n+2)(n+1)$$

long vertices in Γ. Note that for the above equation we exploited that the alternating sum of the binomial coefficients equals zero, that is, $\sum_{k=0}^{n}(-1)^k\binom{n}{k}=0$.

Let x_1, x_2, ..., x_{n+2} be the short vertices of Γ. Γ is locally $\mathbb{W}(B_{n+1})$ at short vertices which implies that for $1\leqslant i\neq j\leqslant n+2$ the common neighborhood $\{x_r\colon\ r\notin\{i,j\}\}^{\perp}$ contains exactly two long vertices which we denote by $y_{i,j}$ and $y_{j,i}$. Since a long vertex is adjacent to exactly n short vertices the $y_{i,j}$ thus defined are all distinct. By construction, $y_{i,j}\perp y_{j,i}$. Further, the $y_{i,j}$ exhaust Y because Γ contains exactly $(n+2)(n+1)$ long vertices. Given two vertices $y_{i,j}$ and $y_{k,l}$, we find $m\in\{1,2,\ldots,n+2\}\backslash\{i,j,k,l\}$ whence $y_{i,j}$ and $y_{k,l}$ are both contained in $x_m^{\perp}\cong\mathbb{W}(B_{n+1})$. $y_{i,j}$ is characterized in $x_m^{\perp}$ as one of the two long vertices contained in $\{x_r\colon\ r\notin\{i,j,m\}\}^{\perp}$. Likewise, $y_{k,l}$ is characterized in $x_m^{\perp}$ as one of the two long vertices contained in $\{x_r\colon\ r\notin\{k,l,m\}\}^{\perp}$. Consequently, for $\{i,j\}\neq\{k,l\}$, $y_{i,j}\perp y_{k,l}$ if and only if $\{i,j\}\cap\{k,l\}=\emptyset$. By Example 3.4, $\Gamma\cong\mathbb{W}(B_{n+2})$. □

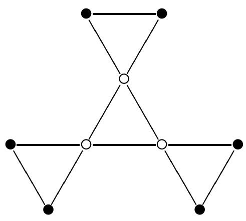
Corollary 4.5. *Let $n\geqslant 4$, and let Γ be a connected bichromatic locally homogeneous graph with $\Delta_s(\Gamma)\cong\mathbb{W}(A_1^s\sqcup C_n)$ and $\Delta_\ell(\Gamma)\cong\mathbb{W}(C_{n+1})$. Then $\Gamma\cong\mathbb{W}(C_{n+2})$.* □

5 Local recognition of $\mathbb{W}(F_4)$

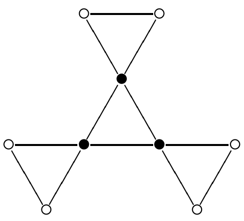
5.1 Graphs locally like $\mathbb{W}(F_4)$

Consider the Weyl graph $\mathbb{W}(F_4)$. Exploiting the description of the crystallographic root system $\Phi(F_4)$ given in Example 3.8 we find that $\mathbb{W}(F_4)$ is a connected bichromatic locally homogeneous graph on 24 vertices with short local graph $\mathbb{W}(B_3)$ and long local graph $\mathbb{W}(C_3)$. Depicting long vertices as filled dots and short vertices as unfilled dots we use the description given in Example 3.4 to draw the local graphs of $\mathbb{W}(F_4)$ as

$\mathbb{W}(B_3)$:



$\mathbb{W}(C_3)$:



We will shortly see that $\mathbb{W}(F_4)$ is not locally recognizable. Before we turn to investigating additional constraints under which we seek to recognize $\mathbb{W}(F_4)$ nonetheless, we study connected bichromatic graphs Γ which are locally like $\mathbb{W}(F_4)$. That is, we study locally homogeneous bichromatic graphs such that $\Delta_s(\Gamma)\cong\mathbb{W}(B_3)$ and $\Delta_\ell(\Gamma)\cong\mathbb{W}(C_3)$. The results we obtain then guide our way in determining appropriate conditions under which we will be (almost) able to recognize $\mathbb{W}(F_4)$. An easy but crucial observation to start with is the following.

Proposition 5.1. *Let Γ be locally like $\mathbb{W}(F_4)$. The short (respectively long) induced subgraph of Γ is isomorphic to a disjoint union of* 4*-cliques.* □

Let Γ be a bichromatic graph that is locally like $\mathbb{W}(F_4)$. Observe that the graph obtained from Γ by exchanging the roles of short and long vertices is locally like $\mathbb{W}(F_4)$ as well. Results that we obtain for short vertices of graphs locally like $\mathbb{W}(F_4)$ are therefore also true for long vertices.

Paraphrasing Proposition 5.1, the vertices of Γ come in 4-cliques of the same type. In order to simplify things it is natural to collapse these 4-cliques into single vertices.

Definition 5.2. *Let Λ be a graph and Π a partition of its vertices. The* contraction Λ/Π *is the graph on Π such that two sets A, $B \in \Pi$ are adjacent whenever there is $a \in A$ and $b \in B$ which are adjacent in Λ. If Λ is bichromatic then Π is required to partition into sets of short and long vertices and Λ/Π is a bichromatic graph in the natural way.*

In this language, we thus investigate the collapsed graph Γ/Π where Π is the partition of Γ into short and long 4-cliques. To this end, we analyze how these 4-cliques relate to each other.

Proposition 5.3. *Let Γ be locally like $\mathbb{W}(F_4)$, and x_1, x_2, x_3, x_4 a short* 4*-clique in Γ. Let $i \neq j$ and $k \neq l$.*

- *$\{x_i, x_j\}^{\perp}$ is locally $K_2^s \sqcup K_2^{\ell}$. In particular, for any pair x_i, x_j there exist unique long vertices $y_{i,j}, y_{j,i}$ contained in $\{x_i, x_j\}^{\perp}$.*
- *$\{x_i, x_j, x_k\}^{\perp}$ contains no long vertex if i, j, k are distinct. In particular, the vertices $y_{i,j}$ are all distinct.*
- *There are exactly* 12 *long vertices adjacent to at least one of the x_i, namely the above vertices $y_{i,j}$.*
- *$y_{i,j} \perp y_{k,l}$ implies that $\{k, l\} = \{i, j\}$ or $\{k, l\} \cap \{i, j\} = \emptyset$.*

Proof. Exploiting the local structure at x_i we see that every short adjacent pair x_i, x_j has exactly two long neighbors in common which we will (arbitrarily) denote by $y_{i,j}$ and $y_{j,i}$. Accordingly, $y_{i,j} \perp y_{j,i}$. Looking at the neighbors of a vertex $y_{i,j}$ reveals that x_i and x_j are the only short vertices among x_1, x_2, x_3, x_4 which are adjacent to $y_{i,j}$. Consequently, the $y_{i,j}$ are 12 distinct vertices. Since three adjacent short vertices share no long neighbors we count that exactly

$$\binom{4}{1}6 - \binom{4}{2}2 = 12$$

long vertices are neighbored to at least one of the vertices x_1, x_2, x_3, x_4. Consequently, the long neighbors of the x_i are precisely the vertices $y_{i,j}$. For the last claim, assume that $y_{i,j} \perp y_{k,l}$ and $\{k, l\} \cap \{i, j\} = \{i_0\}$. A look at the neighbors of x_{i_0} shows that this is a contradiction. □

Proposition 5.4. *Let Γ be locally like $\mathbb{W}(F_4)$, and let Π be the partition of Γ into short and long* 4*-cliques. The contraction Γ/Π has short local graphs each isomorphic to one of $\{\overline{K_n}^{\ell}: n \in \{3, 4, 5, 6\}\}$ and long local graphs each isomorphic to one of $\{\overline{K_n}^{s}: n \in \{3, 4, 5, 6\}\}$.*

Proof. Let $X \in \Gamma/\Pi$ be a short vertex. By Proposition 5.1, the local graph at X is isomorphic to $\overline{K_n}^{\ell}$ for some n. Since the long induced subgraph of the local graph at a short vertex in Γ is isomorphic to $3 \cdot K_2^{\ell}$ we see that $n \geqslant 3$. On the other hand, $X = \{x_1, x_2, x_3, x_4\}$ is a 4-clique of Γ and according to Proposition 5.3 there are 12 long vertices $y_{i,j}$ at distance 1 from X in Γ. Since $y_{i,j}$ and $y_{j,i}$ are adjacent in Γ they are identified in Γ/Π which shows $n \leqslant 6$. ☐

We now do the reverse. Starting with a potential collapse of a graph Γ that is locally like $\mathbb{W}(F_4)$ we try to reconstruct Γ. Since different Γ can have isomorphic collapses this reconstruction necessarily is not unique.

Lemma 5.5. *For every connected bipartite graph* Λ *that is locally* $\overline{K_6}$ *there is a connected bichromatic graph* Γ *that is locally like* $\mathbb{W}(F_4)$ *such that* $\Gamma/\Pi = \Lambda$ *where* Π *is the partition of* Γ *into short and long* 4*-cliques.*

Proof. Let Λ be a bipartite graph that is locally $\overline{K_6}$. Exploiting that Λ is 2-colorable, we may identify Λ with a bichromatic graph such that no two vertices of the same type are adjacent. Accordingly, Λ is locally homogeneous with $\Delta_s(\Lambda) \cong \overline{K_6}^{\ell}$ and $\Delta_\ell(\Lambda) \cong \overline{K_6}^{s}$. For any vertex x of Λ choose a bijection

$$x^{\perp} \to \binom{4}{2}, \quad y \mapsto a(x, y)$$

between its six neighbors and the six 2-subsets of $\{1, 2, 3, 4\}$. To every directed edge (x, y) we thus assigned the 2-subset $a(x, y)$ of $\{1, 2, 3, 4\}$. Construct the bichromatic graph Γ from Λ as follows. For every vertex $x \in \Lambda$ add a 4-clique x_1, x_2, x_3, x_4 of the same type as x to Γ. Then, for $x, y \in \Lambda$ let x_i and y_j be adjacent in Γ if and only if x and y are adjacent in Λ and $(i, j) \in a(x, y) \times a(y, x)$. By construction, contracting the 4-cliques of Γ produces Λ. It is straightforward, and carried out in [Str08, 2.34], to check that Γ is locally like $\mathbb{W}(F_4)$. ☐

The construction in Lemma 5.5 is easily adapted to work for graphs Λ that are locally $\overline{K_3}$, see [Str08, 2.35].

Corollary 5.6. *There exist infinitely many finite connected bichromatic graphs that are locally like* $\mathbb{W}(F_4)$.

Proof. We claim that there are infinitely many finite connected bipartite graphs Λ that are locally $\overline{K_6}$. To this end, note that the graphs $C_k \times C_m \times C_n$ are connected and locally $\overline{K_6}$ for $k, m, n \geqslant 4$, cf. Remark 2.2. Since cycles C_n are 2-colorable whenever n is even, the graphs $C_k \times C_m \times C_n$ are 2-colorable and hence bipartite whenever k, m, n are all even. The claim follows from Lemma 5.5. ☐

5.2 Recognition results

We now study further properties of the Weyl graph $\mathbb{W}(F_4)$ in order to characterize $\mathbb{W}(F_4)$ among the connected bichromatic graphs that are locally like $\mathbb{W}(F_4)$. For more details we refer to the thesis [Str08] of the third author. We start with some easy observations.

Proposition 5.7. *Let* Γ *be a finite bichromatic graph that is locally like* $\mathbb{W}(F_4)$. *Then the numbers of short and long vertices in* Γ *are the same.* ☐

Corollary 5.8. *Let* Γ *be a finite bichromatic graph that is locally like* $\mathbb{W}(F_4)$*. Then* $|\Gamma|$ *is divisible by* 8 *and* $|\Gamma| \geqslant 24$*.* ☐

Since $|\mathbb{W}(F_4)| = 24$ we see that, in a sense, $\mathbb{W}(F_4)$ is maximally tight among the graphs that are locally like $\mathbb{W}(F_4)$. There are several further properties of a graph, for instance its diameter, that describe its tightness. The following notion of tight connectedness is another way to express tightness of a bichromatic graph.

Definition 5.9. *A bichromatic graph is said to be* tightly connected *if every long vertex has a neighbor in every short component and vice versa.*

These three notions of tightness, however, are not local in nature (where a local property is meant to be one which can be expressed in terms of the neighbors of each vertex), and in order to find a more local notion to describe the tightness of $\mathbb{W}(F_4)$ we investigate the relation of vertices at distance 2.

Proposition 5.10. *Let* Γ *be locally like* $\mathbb{W}(F_4)$*, and let* $x, y \in \Gamma$ *be at distance* 2*.*

- *If* x, y *are both short (respectively long) vertices then* $\{x, y\}^{\perp} \cong \mu(x, y) \cdot K_1^{\ell}$ *(respectively* K_1^s*) for some* $\mu(x, y) \in \{1, 2, 3\}$*.*
- *If* x*,* y *are of mixed type then* $\{x, y\}^{\perp} \cong \mu_s(x, y) \cdot K_2^s \sqcup \mu_{\ell}(x, y) \cdot K_2^{\ell}$ *for some* $\mu_s(x, y)$*,* $\mu_{\ell}(x, y) \in \{0, 1\}$*.*

Proof. First, let x, y be two short vertices at distance 2. Since short vertices come in 4-cliques the common neighbors $\{x, y\}^{\perp}$ of x, y can only contain long vertices. The structure of the local graph at such a long vertex z implies that $\{x, y, z\}^{\perp}$ is empty. Accordingly, $\{x, y\}^{\perp}$ is a coclique on long vertices. Set $\mu(x, y) = |\{x, y\}^{\perp}|$. Looking at $x^{\perp}$, we find that $\mu(x, y) \in \{1, 2, 3\}$.

Likewise, let x be a short and y a long vertex at distance 2. Let $z \in \{x, y\}^{\perp}$ be a, say, short vertex. By looking at the neighbors of z we see that $\{x, y, z\}^{\perp}$ contains exactly one more short vertex. Hence, $\{x, y\}^{\perp}$ is a disjoint sum $\mu_s(x, y) \cdot K_2^s \sqcup \mu_{\ell}(x, y) \cdot K_2^{\ell}$ of pairs of short vertices and pairs of long vertices for some $\mu_s(x, y)$, $\mu_{\ell}(x, y)$. Looking at $x^{\perp}$ and $y^{\perp}$, we further see that $\mu_s(x, y), \mu_{\ell}(x, y) \in \{0, 1\}$. ☐

For the Weyl graph $\mathbb{W}(F_4)$ the parameters μ, μ_s, μ_{ℓ} defined in Proposition 5.10 are constant and take the maximum possible values $\mu = 3$ and $\mu_s = \mu_{\ell} = 1$ which is another, more local, instantiation of the tightness of $\mathbb{W}(F_4)$. The following fact relates the parameters μ_s and μ_{ℓ} of Γ to the local structure of the contraction Γ/Π studied in Proposition 5.4.

Proposition 5.11. *Let* Γ *be locally like* $\mathbb{W}(F_4)$*, and let* Π *be the partition of* Γ *into* 4*-cliques.*

- $\mu_s = \mu_{\ell} = 1$ *if and only if the contraction* Γ/Π *is locally homogeneous with* $\Delta_s(\Gamma/\Pi) \cong \overline{K_3}^{\ell}$ *and* $\Delta_{\ell}(\Gamma/\Pi) \cong \overline{K_3}^{s}$*.*
- $\mu_s + \mu_{\ell} = 1$ *if and only if the contraction* Γ/Π *is locally homogeneous with* $\Delta_s(\Gamma/\Pi) \cong \overline{K_6}^{\ell}$ *and* $\Delta_{\ell}(\Gamma/\Pi) \cong \overline{K_6}^{s}$*.*

Proof. Suppose that $\mu_s = \mu_\ell = 1$. Let $X = \{x_1, x_2, x_3, x_4\}$ be a short vertex of Γ/Π. Adopting the notation of Proposition 5.3, let $y_{i,j}$ and $y_{j,i}$ be the long vertices adjacent to both x_i and x_j. Let $i \notin \{k, l\}$ so that x_i and $y_{k,l}$ are at distance 2, and let j be the index such that $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Since $\mu_\ell(x_i, y_{k,l}) = 1$, there are two long vertices adjacent to both x_i and $y_{k,l}$. According to Proposition 5.3 the only possibilities are $y_{i,j}$ and $y_{j,i}$. Therefore $\{y_{i,j}, y_{j,i}, y_{k,l}, y_{l,k}\}$ is a long vertex of Γ/Π and Proposition 5.3 implies that the local graph at X is isomorphic to $\overline{K_3}^\ell$. Analogously for long local graphs.

On the other hand, assume that Γ/Π is locally homogeneous with $\Delta_s(\Gamma/\Pi) \cong \overline{K_3}^\ell$ and $\Delta_\ell(\Gamma/\Pi) \cong \overline{K_3}^s$. Let x be a short and y be a long vertex of Γ which are at distance 2. Observe that x and y are contained in adjacent equivalence classes of Γ/Π. Let $X = \{x_1, x_2, x_3, x_4\}$ be the short vertex of Γ/Π where $x = x_i$ for some i. By assumption, X is adjacent in Γ/Π to exactly three long vertices which by Proposition 5.3 are of the form $\{y_{i,j}, y_{j,i}, y_{k,l}, y_{l,k}\}$ for $\{i, j, k, l\} = \{1, 2, 3, 4\}$. y therefore equals $y_{k,l}$ for some $\{k, l\} \not\ni i$. Choose j as above. Then

$$\{x, y\}^\perp = \{x_i, y_{k,l}\}^\perp = \{x_k, x_l, y_{i,j}, y_{j,i}\},$$

and therefore $\mu_s(x, y) = \mu_\ell(x, y) = 1$.

The second equivalence is proved similarly. □

The following theorem summarizes our recognition results for the Weyl graph $\mathbb{W}(F_4)$. Note that all of the provided conditions under which a graph Γ is almost recognized as $\mathbb{W}(F_4)$ are statements which describe the tightness of Γ. We denote with Γ_{24b} the graph on 24 vertices constructed in course of the proof of Proposition 5.13.

Theorem 5.12. *Let Γ be a connected bichromatic graph that is locally like $\mathbb{W}(F_4)$. Assume that*

- $|\Gamma| = 24$, *or*
- Γ *is tightly connected, or*
- Γ *has diameter* 2, *or*
- $\mu = 3$.

If one of these conditions holds then Γ is isomorphic to $\mathbb{W}(F_4)$ or to Γ_{24b}. In particular, $\mathrm{Aut}(\Gamma) \cong W(F_4)/Z$ *where Z denotes the center of $W(F_4)$.*

We prove Theorem 5.12 by a series of propositions.

Proposition 5.13. *Let Γ be a connected bichromatic graph that is locally like $\mathbb{W}(F_4)$. If $|\Gamma| = 24$ then $\Gamma \cong \mathbb{W}(F_4)$ or $\Gamma \cong \Gamma_{24b}$. Further,* $\mathrm{Aut}(\Gamma) \cong W(F_4)/Z$.

Proof. As observed in Corollary 5.8, every graph that is locally like $\mathbb{W}(F_4)$ has at least 12 short and 12 long vertices. Γ therefore consists of exactly 12 vertices of each type.

Let x_1, x_2, x_3, x_4 be a short 4-clique. Adopting the notation of Proposition 5.3, let $y_{i,j}$ and $y_{j,i}$ be the long vertices adjacent to both x_i and x_j. The $y_{i,j}$ are 12 distinct vertices and therefore constitute the long vertices of Γ. It follows from Proposition 5.3 that the three long 4-cliques are given by $y_{i,j}, y_{j,i}, y_{k,l}, y_{l,k}$ for disjoint $\{i, j\}$ and $\{k, l\}$.

Each of the remaining eight short vertices has exactly two long neighbors in each of the three long 4-cliques. Let x_5 be one of remaining short vertices. The two neighbors of x_5 in a 4-clique $y_{i,j}$, $y_{j,i}$, $y_{k,l}$, $y_{l,k}$ are one of $y_{i,j}$, $y_{j,i}$ along with one of $y_{k,l}$, $y_{l,k}$. We ambiguously defined the vertices $y_{i,j}$, $y_{j,i}$ as the long vertices contained in $\{x_i,\ x_j\}^{\perp}$ so we may as well assume that x_5 is adjacent to $y_{i,j}$ and $y_{k,l}$ with $i < j$ and $k < l$. Let x_6 be the unique short vertex also adjacent to $y_{1,2}$, $y_{3,4}$. Likewise, let x_7 be the short vertex also adjacent to $y_{1,3}$, $y_{2,4}$, and x_8 the short vertex also adjacent to $y_{1,4}$, $y_{2,3}$. By construction, x_5, x_6, x_7, x_8 is a 4-clique. Notice that for instance $x_5, x_6 \in \{y_{1,2},\ y_{3,4}\}^{\perp}$ implies that $x_7, x_8 \in \{y_{2,1},\ y_{4,3}\}^{\perp}$. Altogether this determines the induced subgraph on $x_1, x_2, \dots, x_8$ along with the vertices $y_{i,j}$.

Let $x_9, x_{10}, x_{11}, x_{12}$ be the remaining short 4-clique. We may assume that x_9, x_{10} are the short vertices contained in $\{y_{1,2},\ y_{4,3}\}^{\perp}$. Accordingly, $x_{11},\ x_{12} \in \{y_{2,1},\ y_{3,4}\}^{\perp}$. We may also assume that x_9 is contained in $\{y_{1,3},\ y_{4,2}\}^{\perp}$ (because if both x_9 and x_{10} were not contained in $\{y_{1,3},\ y_{4,2}\}^{\perp}$ then both $x_{11}, x_{12} \in \{y_{1,3},\ y_{4,2}\}^{\perp}$ which contradicts $x_{11}, x_{12} \in \{y_{2,1},\ y_{3,4}\}^{\perp}$). Further, we may assume that x_{11} is the second short vertex contained in $\{y_{1,3},\ y_{4,2}\}^{\perp}$. Consider the two short vertices in $\{y_{1,4},\ y_{3,2}\}^{\perp}$. These can be either x_9, x_{12} or x_{10}, x_{11}, and either choice determines Γ. Denote with Γ_{24a} the graph corresponding to the choice $x_9,\ x_{12} \in \{y_{1,4},\ y_{3,2}\}^{\perp}$, and with Γ_{24b} the graph corresponding to the choice $x_{10},\ x_{11} \in \{y_{1,4},\ y_{3,2}\}^{\perp}$. The following table summarizes adjacency involving the vertices $x_9, x_{10}, x_{11}, x_{12}$.

by construction	$x_9, x_{10} \perp y_{1,2}, y_{4,3}$	$x_{11}, x_{12} \perp y_{2,1}, y_{3,4}$
	$x_9, x_{11} \perp y_{1,3}, y_{4,2}$	$x_{10}, x_{12} \perp y_{3,1}, y_{2,4}$
Γ_{24a}	$x_9, x_{12} \perp y_{1,4}, y_{3,2}$	$x_{10}, x_{11} \perp y_{4,1}, y_{2,3}$
Γ_{24b}	$x_9, x_{12} \perp y_{4,1}, y_{2,3}$	$x_{10}, x_{11} \perp y_{1,4}, y_{3,2}$

An implementation in the computer algebra system SAGE, see [SAG07], of the graphs Γ_{24a} and Γ_{24b} can be found in the appendix of [Str08]. In particular, it is verified that Γ_{24a} and Γ_{24b} are nonisomorphic, that the automorphism group of both graphs is isomorphic to $W(F_4)/Z$, and that Γ_{24a} is isomorphic to $\mathbb{W}(F_4)$. □

Proposition 5.14. *Let Γ be a connected bichromatic graph that is locally like $\mathbb{W}(F_4)$. If Γ is tightly connected then $\Gamma \cong \mathbb{W}(F_4)$ or $\Gamma \cong \Gamma_{24b}$.*

Proof. Fix a short 4-clique x_1, x_2, x_3, x_4. Because of tightness every long vertex is adjacent to one of the x_i, and by Proposition 5.3 there are exactly 12 such long vertices. Thus Γ consists of 12 long vertices. Likewise, Γ contains exactly 12 short vertices. Hence, $|\Gamma| = 24$, and the claim follows from Proposition 5.13. □

Proposition 5.15. *Let Γ be a connected bichromatic graph that is locally like $\mathbb{W}(F_4)$. If Γ has diameter 2 then $\Gamma \cong \mathbb{W}(F_4)$ or $\Gamma \cong \Gamma_{24b}$.*

Proof. Let x_1, x_2, x_3, x_4 be a short 4-clique. As in Proposition 5.3 let $y_{i,j}, y_{j,i}$ be the long vertices adjacent to both x_i and x_j. Assume that there is a long vertex v which is not among the 12 long vertices $y_{i,j}$. Because v is not adjacent to any of the x_i and since the diameter of Γ is 2, we find a long vertex that connects x_1 and v. Without loss of generality let this long vertex be $y_{1,2}$. This prevents $y_{1,2}$, $y_{2,1}$, $y_{3,4}$, $y_{4,3}$ from forming a long 4-clique. By Proposition 5.3 there are thus long vertices v_1, v_2 not among the $y_{i,j}$ such that $y_{3,4}$, $y_{4,3}$, v_1, v_2 form a long 4-clique. Again, v_1 is not adjacent to any of the x_i and hence is connected to x_1 by a long vertex. This is a contradiction since the long vertices adjacent to x_1 are the vertices $y_{1,j}$, $y_{j,1}$.

Consequently, Γ contains no further long vertices besides the 12 vertices $y_{i,j}$. The same reasoning reveals that Γ contains exactly 12 short vertices. We conclude that $|\Gamma| = 24$. According to Proposition 5.13 this proves that $\Gamma \cong \mathbb{W}(F_4)$ or $\Gamma \cong \Gamma_{24b}$. □

Proposition 5.16. *Let Γ be a connected bichromatic graph that is locally like $\mathbb{W}(F_4)$. Let x_1 be a short vertex of Γ. If for any nonadjacent pair of long vertices y, $y' \in x_1^{\perp}$ we have $\mu(y, y') = 3$ and for any nonadjacent pair of a short and a long vertex $x, y \in x_1^{\perp}$ we have $\mu_{\ell}(x, y) = 1$ then $\Gamma \cong \mathbb{W}(F_4)$ or $\Gamma \cong \Gamma_{24b}$.*

Proof. Let x_1, x_2, x_3, x_4 be the short 4-clique containing x_1. As in Proposition 5.3 denote with $y_{i,j}$, $y_{j,i}$ the long vertices adjacent to both x_i and x_j. The long vertex $y_{1,2}$ and the short vertex x_3 are nonadjacent neighbors of x_1. Since $\mu_{\ell}(x_3, y_{1,2}) = 1$ we find two long vertices in $\{x_3, y_{1,2}\}^{\perp}$. By Proposition 5.3 the only candidates are $y_{3,4}$ and $y_{4,3}$. Consequently, $y_{1,2}$, $y_{2,1}$, $y_{3,4}$, $y_{4,3}$ form a long 4-clique. Likewise, by considering the nonadjacent pairs $y_{1,3}$, x_4 respectively $y_{1,4}$, x_2 we find that $y_{1,3}$, $y_{3,1}$, $y_{2,4}$, $y_{4,2}$ respectively $y_{1,4}$, $y_{4,1}$, $y_{2,3}$, $y_{3,2}$ form a long 4-clique.

Consider the long 4-clique $y_{1,2}$, $y_{2,1}$, $y_{3,4}$, $y_{4,3}$. It follows from Proposition 5.3 that there are 8 short neighbors of this clique besides x_1, x_2, x_3, x_4. Denote these neighbors by $x_5, x_6 \ldots, x_{12}$. We may assume that

$$\{x_5, x_6, y_{1,2}, y_{3,4}\}, \quad \{x_7, x_8, y_{2,1}, y_{4,3}\}, \quad \{x_9, x_{10}, y_{1,2}, y_{4,3}\}, \quad \{x_{11}, x_{12}, y_{2,1}, y_{3,4}\}$$

form mixed 4-cliques. Looking at the neighborhood of $y_{1,2}$ we see that x_5 is not adjacent to x_{10}. Likewise, x_5 is not adjacent to x_{12}.

Now, consider the long vertex $y_{1,3}$. By assumption $\{y_{1,2}, y_{1,3}\}^{\perp}$ contains 2 short vertices besides x_1. Since these two short vertices are not adjacent we may assume them to be x_5 and x_9. Likewise, $y_{2,1}$ and $y_{1,3}$ share two short neighbors besides x_1 which we may assume to be x_7 and x_{11}. Summarizing, $y_{1,3}$ is adjacent to x_5, x_7, x_9 and x_{11}. Because $\{y_{1,3}, y_{3,1}\}^{\perp}$ contains no short vertex besides x_1 and x_3, we analogously find that $y_{3,1}$ is adjacent to x_6, x_8, x_{10} and x_{12}. Because the short neighbors of $y_{1,3}$ and $y_{3,1}$ come in adjacent pairs we deduce that x_5, x_6, x_7, x_8 as well as $x_9, x_{10}, x_{11}, x_{12}$ form a short 4-clique.

In particular, we just showed that the six short neighbors of $y_{1,3}$ are found among x_1, x_2, ..., x_{12}. The same argument applies analogously to the other $y_{1,j}$ and $y_{i,1}$. Consider for instance the vertex $y_{2,4}$. Exploiting that $y_{2,4}$ is adjacent to both $y_{1,3}$ and $y_{3,1}$, and thus shares two short neighbors with each, we find that the six short neighbors of $y_{2,4}$ are among the x_i. Accordingly, we conclude that any short vertex adjacent to one of the 12 long vertices of the form $y_{i,j}$ is among the short vertices $x_1, x_2, \ldots, x_{12}$. By reciprocity as in Proposition 5.7 we find that all long neighbors of one of the 12 short vertices x_1, x_2, ..., x_{12} are among the $y_{i,j}$. Since all of these vertices came in 4-cliques of the same type the connectedness of Γ implies that $|\Gamma| = 24$. The claim follows by Proposition 5.13. □

Corollary 5.17. *Let Γ be a connected bichromatic graph that is locally like $\mathbb{W}(F_4)$. If $\mu = 3$ and $\mu_s = \mu_{\ell} = 1$ then $\Gamma \cong \mathbb{W}(F_4)$ or $\Gamma \cong \Gamma_{24b}$.* □

Proposition 5.18. *Let Γ be a bichromatic graph that is locally like $\mathbb{W}(F_4)$. If $\mu = 3$ then $\mu_s = \mu_{\ell} = 1$.*

Proof. Let Π be the partition of Γ into 4-cliques of the same type, and let $X = \{x_1, x_2, x_3, x_4\}$ be a short vertex of Γ/Π. As in Proposition 5.3 denote with $y_{i,j}, y_{j,i}$ the long vertices adjacent to both x_i and x_j. Recall that $y_{i,j}$ and $y_{k,l}$ are at distance 2 if $|\{i,j\} \cap \{k,l\}| = 1$. Note that a long vertex has six short neighbors which come in adjacent pairs. Further, because $\mu = 3$, two long vertices at distance 2 share three short neighbors. Therefore, if y is a long vertex neighbored to the adjacent pair of short vertices x, x' and if y' is a long vertex at distance 2 from y then y' is neighbored to exactly one x, x'.

Denote with z_1, z_2 a pair of adjacent short vertices besides x_1, x_2 neighbored to $y_{1,2}$. The vertices $y_{1,2}$ and $y_{1,3}$ are at distance 2. Without loss we may therefore assume that $y_{1,3}$ is adjacent to z_1. Let z_3 be the unique short vertex adjacent to z_1 and $y_{1,3}$, and let z_4 be the unique short vertex such that z_1, z_2, z_3, z_4 form a short 4-clique. Let y be a long vertex at distance 2 from both $y_{1,2}$ and $y_{1,3}$. Then y is either adjacent to both z_1 and z_4 or y is adjacent to both z_2 and z_3. The vertices $y_{1,4}$ and $y_{4,1}$ are each long vertices at distance 2 from both $y_{1,2}$ and $y_{1,3}$. $y_{1,4}$ and $y_{4,1}$ are adjacent and thus share no short neighbors besides x_1 and x_4. Accordingly, we may assume that $y_{1,4}$ is adjacent to both z_1 and z_4, and that $y_{4,1}$ is adjacent to both z_2 and z_3. The following table summarizes the situation.

$$\begin{array}{ll} y_{1,2} \perp z_1, z_2 & y_{2,1} \perp z_3, z_4 \\ y_{1,3} \perp z_1, z_3 & y_{3,1} \perp z_2, z_4 \\ y_{1,4} \perp z_1, z_4 & y_{4,1} \perp z_2, z_3 \end{array}$$

Consider the vertex $y_{2,3}$. $y_{2,3}$ has distance 2 from both $y_{1,2}$ and $y_{1,3}$. Therefore either $y_{2,3} \perp z_1, z_4$ or $y_{2,3} \perp z_2, z_3$. In the former case, $y_{2,3}, y_{1,4} \in \{z_1, z_4\}^{\perp}$. Since z_1 and z_4 are adjacent the vertices $y_{2,3}$ and $y_{1,4}$ are adjacent as well. Likewise, the latter case implies that $y_{2,3}$ and $y_{4,1}$ are adjacent. In both cases, the long vertices $y_{2,3}, y_{3,2}, y_{1,4}, y_{4,1}$ form a 4-clique.

Analogously, one shows that the vertices $y_{i,j}, y_{j,i}, y_{k,l}, y_{l,k}$ form a 4-clique whenever the index sets $\{i,j\}$ and $\{k,l\}$ are disjoint. The local graph at X in Γ/Π therefore is isomorphic to $\overline{K_3}^{\ell}$. Since X was an arbitrary short vertex, and since the same argument works for long vertices, we proved that Γ/Π is locally homogeneous with $\Delta_s(\Gamma/\Pi) \cong \overline{K_3}^{\ell}$ and $\Delta_{\ell}(\Gamma/\Pi) \cong \overline{K_3}^{s}$. It follows from Proposition 5.11 that $\mu_s = \mu_{\ell} = 1$. □

Corollary 5.19. *Let Γ be a connected bichromatic graph that is locally like $\mathbb{W}(F_4)$. If $\mu = 3$ then $\Gamma \cong \mathbb{W}(F_4)$ or $\Gamma \cong \Gamma_{24b}$.* □

In Proposition 5.10 we associated the parameters μ, μ_s, μ_{ℓ} to a graph that is locally like $\mathbb{W}(F_4)$. In light of Theorem 5.12 and Corollary 5.17 one is interested in conditions under which $\mu_s + \mu_{\ell}$ respectively μ are constant. We only record the following observation here and refer to [Str08] for more conditions.

Proposition 5.20. *Let Γ be locally like $\mathbb{W}(F_4)$ and transitive on 3-paths of the same type. Then μ is constant on pairs of short vertices as well as on pairs of long vertices.*

Proof. Recall that μ takes values in $\{1, 2, 3\}$. Let x_1, x_2 be nonadjacent short vertices at distance 2 such that μ restricted to pairs of short vertices is maximal. Let $y \in \{x_1, x_2\}^{\perp}$. Consider another nonadjacent pair of short vertices x_1', x_2' at distance 2 and let $y' \in \{x_1', x_2'\}^{\perp}$. y and y' are both long whence by assumption we find an automorphism $\psi \in \mathrm{Aut}(\Gamma)$ such that $\psi(x_1, y, x_2) = (x_1', y', x_2')$ or $\psi(x_1, y, x_2) = (x_2', y', x_1')$. Since $\{x_1', x_2'\}^{\perp} = \psi(\{x_1, x_2\}^{\perp})$ this proves that μ is constant on pairs of short vertices. Likewise for pairs of long vertices. □

Remark 5.21. In Corollary 5.17 we recognized $\mathbb{W}(F_4)$ and Γ_{24b} as the only connected bichromatic graphs that are locally like $\mathbb{W}(F_4)$ and for which $\mu = 3$ and $\mu_s = \mu_\ell = 1$ where μ, μ_s, μ_ℓ are the parameters introduced in Proposition 5.10. This characterizes these two graphs as the tightest graphs that are locally like $\mathbb{W}(F_4)$. Furthermore, we showed in Proposition 5.18 that there is no graph locally like $\mathbb{W}(F_4)$ for which $\mu = 3$ and $\mu_s + \mu_\ell = 1$. The next level of tightness in terms of the parameters μ, μ_s, μ_ℓ is therefore achieved by graphs with $\mu = 2$ and $\mu_s = \mu_\ell = 1$. In [Str08] two particularly symmetric graphs locally like $\mathbb{W}(F_4)$ are characterized for which $\mu = 2$ and $\mu_s = \mu_\ell = 1$. Both are graphs on 32 vertices which in view of Corollary 5.8 is the next smallest number (after 24) of vertices for a graph that is locally like $\mathbb{W}(F_4)$.

6 Group theoretic applications

Let G be a group acting on a graph Γ. In this section we explore how a local recognition result for the graph Γ may be turned into a group theoretical statement about G. In fact, what we seek to do is to encode the recognition of the local structure of Γ into a statement about the local structure of G. For a more detailed exposition we refer to the third author's thesis [Str08].

The following observations are stated in a slightly more general setting than needed right away for the local recognition of the symmetric groups so that they will be of use as well when we generalize to bichromatic graphs.

Proposition 6.1. *Let G be a group, $x, y \in G$, and Γ a graph with vertices the conjugates of x and y such that $\{x, y\}^G \subset E(\Gamma)$.*

- *If $G = \langle C_G(x), C_G(y)\rangle$ then Γ is connected.*
- *If $G = \langle x^G, y^G\rangle$ and G acts on Γ by conjugation then the kernel of this action is $Z(G)$.*

Proof. For the first claim, suppose that $G = \langle C_G(x), C_G(y)\rangle$. Denote with Λ the connected component of Γ containing x and y. Let $g \in G$ such that x^g and y^g are contained in Λ. Let $h \in C_G(x)$. Consequently, $x^{hg} = x^g \in \Lambda$. Moreover,

$$\{x, y\}^{hg} = \{x^g, y^{hg}\}$$

which implies that $y^{hg} \perp x^g$ in Γ. Hence y^{hg} is contained in Λ as well. The same argument applies to $h \in C_G(y)$. By assumption, any $g \in G$ can be written as $g = h_1 h_2 \cdots h_n$ with each h_i contained either in $C_G(x)$ or $C_G(y)$. Connectedness of Γ follows by induction.

Now, assume that $G = \langle x^G, y^G\rangle$ and that G acts on Γ by conjugation. Clearly, $Z(G)$ is contained in the kernel of the action. On the other hand, let $g \in G$ be an element acting trivially. This means that g centralizes the conjugates of x and y. But these generate G and the claim follows. □

6.1 Recognizing symmetric groups

The following theorem provides a local characterization of the symmetric groups Sym_n. It is the paradigmatic example given in [GLS94, Theorem 27.1] to illustrate the strategy of recognizing a group from local information based on centralizers of involutions. The proof given here is based on the local recognition of Kneser graphs stated in Theorem 2.5. Actually, it is this characterization of the symmetric groups that was a motivation for the second author to pursue the local recognition of Kneser graphs, see [Hal87].

Theorem 6.2. *Let $n \geqslant 7$, and let G be a group with involutions x, y such that*

- $C_G(x) = \langle x \rangle \times J$ *with* $J \cong \mathrm{Sym}_n$,
- $C_G(y) = \langle y \rangle \times K$ *with* $K \cong \mathrm{Sym}_n$,
- x *is a transposition in* K,
- y *is a transposition in* J,
- $J \cap K$ *contains an involution* z *that is a transposition in both* J *and* K.

If $G = \langle J, K \rangle$, *then* $G \cong \mathrm{Sym}_{n+2}$.

Proof. y and z are both transpositions in J and hence conjugate by an involution $u \in J$. Accordingly, $(x, y, z)^u = (x, z, y)$. Likewise, x and z are conjugate by an involution $v \in K$. We conclude that x, y, z are all conjugate in G, and that $\langle u, v \rangle$ acts by conjugation on the set $\{x, y, z\}$ as Sym_3. In particular, we find w such that $(y, x) = (x, y)^w$. Let Γ be the graph on the G-conjugates of x with edges $\{x, y\}^G$. Note that two vertices a, b in Γ are adjacent if and only if $(a, b) = (x, y)^g$ for some $g \in G$. By Proposition 6.1 the graph Γ is connected. The neighbors of x are the J-conjugates of y since

$$x^{\perp} = \{y^g \colon\ g \in C_G(x)\} = \{y^g \colon\ g \in J\}.$$

By construction, $\{x, y, z\}$ is a triangle of Γ. We claim that G is transitive on the oriented triangles of Γ. Indeed, let (a, b, c) be an oriented triangle of Γ. By edge-transitivity we find $g \in G$ such that $(a, b) = (x, y)^g$. Set $d = z^g$. Because b, c, d are all neighbors of a they are J^g-conjugates in J^g. Consequently, b, c, d are transpositions in $J^g \cong \mathrm{Sym}_n$. Since $[b, c] = [b, d] = 1$ we find $h \in J^g$ such that $(b, c) = (b, d)^h$. Thus $(a, b, c) = (a, b, d)^h$ which shows that (a, b, c) is indeed conjugate to (x, y, z).

We observed that the neighbors of x are the J-conjugates of y which are exactly the transpositions of J. Two neighbors a, b of x are adjacent if and only if (x, a, b) is a triangle in Γ. By the transitivity on triangles, this is the case if and only if we find $g \in G$ such that $(x, a, b) = (x, y, z)^g$ or, equivalently, if and only if we find $g \in J$ such that $(a, b) = (y, z)^g$. By assumption, y, z are commuting transpositions of $J \cong \mathrm{Sym}_n$. But two transpositions a, b in J are conjugate to the two commuting transpositions y, z if and only if they commute themselves. Since this is the case precisely when a, b have disjoint support we see that $x^{\perp}$ is isomorphic to the Kneser graph $K(n, 2)$. By vertex-transitivity it follows that Γ is locally $K(n, 2)$. Since $n \geqslant 7$ and Γ is connected Theorem 2.5 implies that $\Gamma \cong K(n+2, 2)$.

The J-conjugates of y generate J, and likewise the K-conjugates of x generate K. The conjugates of x thus generate $G = \langle J, K \rangle$ and Proposition 6.1 implies that $G/Z(G)$ acts faithfully on Γ. Note that $Z(G) \leqslant C_G(x)$ and hence $Z(G) \leqslant Z(C_G(x)) = \langle x \rangle$. Since $x \notin Z(G)$ we find that the center of G is trivial. Consequently, G identifies with a subgroup of $\mathrm{Aut}(\Gamma)$. It is a consequence of the Erdös-Ko-Rado theorem, see [EKR61], that $\mathrm{Aut}(\Gamma)$ is isomorphic to Sym_{n+2}. Since G acts transitively on Γ the orbit-stabilizer formula finally implies that in fact $G \cong \mathrm{Sym}_{n+2}$. ☐

Since the symmetric group Sym_n is a Coxeter group of type A_{n-1}, Theorem 6.2 can be paraphrased as a recognition result for Coxeter groups of type A_n.

Theorem 6.3. *Let $n \geqslant 6$, and let G be a group with involutions x, y such that*

- $C_G(x) = \langle x \rangle \times J$ *with* $J \cong W(A_n)$,
- $C_G(y) = \langle y \rangle \times K$ *with* $K \cong W(A_n)$,
- x *is a reflection in* K,
- y *is a reflection in* J,
- $J \cap K$ *contains an involution* z *that is a reflection in both* J *and* K.

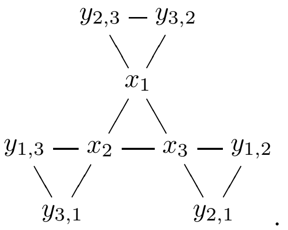
If $G = \langle J, K \rangle$, *then* $G \cong W(A_{n+2})$. □

6.2 Recognizing $W(F_4)$

Observe that the local recognition result of $W(A_n)$ stated in Theorem 6.3 was based on the local recognition of the corresponding Weyl graph $\mathbb{W}(A_n)$. In the sequel, we wish to apply our local recognition results for $\mathbb{W}(F_4)$ in a similar way to recognize the group $W(F_4)$.

If x is a short root reflection of $W(F_4)$ then its centralizer is $\langle x \rangle \times W(B_3)$. Note that the fact that the centralizer is a Coxeter group itself is not surprising in view of Remark 3.2. Accordingly, we take a closer look at $W(B_3)$ in the following example. Of course, analogous statements are true for $W(C_3)$ which occurs in the centralizer of a long root reflection.

Example 6.4. Recall the description of the Weyl graph $\mathbb{W}(B_3)$ in Example 3.4. $W(B_3)$ contains three short root reflections x_1, x_2, x_3 and six long root reflections $y_{i,j}$, $1 \leqslant i \neq j \leqslant 3$. The corresponding Weyl graph is



By definition, the order of the product $x_i\, y_{j,k}$ is 4 if x_i and $y_{j,k}$ do not commute or, equivalently, if $i \in \{j, k\}$. We therefore deduce that ${y_{j,k}}^{x_j} = {y_{j,k}}^{x_k} = y_{k,j}$ and ${x_j}^{y_{j,k}} = {x_j}^{y_{k,j}} = x_k$. In other words, if x and y are a noncommuting short and long root reflection then y^x is the unique long root reflection commuting with y and x^y is the unique short root reflection besides x not commuting with y. Consider two nonadjacent long vertices, say $y_{1,2}$ and $y_{1,3}$. Then $(y_{1,2}\, y_{1,3})^3 = 1$, and we deduce that ${y_{1,2}}^{y_{1,3}} = {y_{1,3}}^{y_{1,2}}$ does not commute with $y_{1,2}$ or $y_{1,3}$. Hence ${y_{1,2}}^{y_{1,3}}$ equals either $y_{2,3}$ or $y_{3,2}$.

Let $W(B_3)$ act on $\mathbb{W}(B_3)$ by conjugation. The above considerations show that the product $x_1\, x_2\, x_3$ acts trivially. On the other hand, we observe that $W(B_3)$ acts transitively on ordered pairs of nonadjacent vertices of the same type.

Before attempting to recognize a group G as the Coxeter group $W(F_4)$ we present conditions under which we can construct a graph from G that is locally like $\mathbb{W}(F_4)$.

Proposition 6.5. *Let G be a group with nonconjugate involutions x, y such that*

- $C_G(x) = \langle x \rangle \times J$ *with* $J \cong W(B_3)$*,*
- $C_G(y) = \langle y \rangle \times K$ *with* $K \cong W(C_3)$*,*
- x *(respectively y) is a short (respectively long) root reflection in K (respectively J),*
- $J \cap K$ *contains involutions x^u, y^v where $u, v \in G$ such that x^u (respectively y^v) is a short (respectively long) root reflection in K as well as in J, and*
- $J \cap J^u$ *(respectively $K \cap K^v$) contains an involution that is a short (respectively long) root reflection in both J and J^u (respectively K and K^v).*

Let Γ be the bichromatic graph with short vertices the conjugates of x and with long vertices the conjugates of y, and edges

$$E(\Gamma) = \{x, x^u\}^G \cup \{y, y^v\}^G \cup \{x, y\}^G.$$

Then Γ is locally like $\mathbb{W}(F_4)$. Furthermore, if $G = \langle J, K \rangle$ then Γ is connected.

Proof. Set $x_1 = x^u$ and $y_1 = y^v$. Further, let x_2 (respectively y_2) be the involution that is a short (respectively long) root reflection in both J and J^u (respectively K and K^v). We have x_1, x_2, y, $y_1 \in J \cong W(B_3)$ where x_1 is a short root reflection, and y, y_1 two commuting long root reflections. Exploiting that x_1 commutes with both y and y_1, it follows from Example 6.4 that $y_1 = y^{x_2}$. Likewise, $x_1 = x^{y_2}$.

Let Γ be the bichromatic graph with short vertices the conjugates of x and with long vertices the conjugates of y, and edges

$$E(\Gamma) = \{x, x_1\}^G \cup \{y, y_1\}^G \cup \{x, y\}^G.$$

Note that two short (respectively two long, respectively one short and one long) vertices a, b are adjacent in Γ if and only if $(a, b) = (x, x_1)^g$ (respectively $(a, b) = (y, y_1)^g$, respectively $(a, b) = (x, y)^g$) for some $g \in G$. It follows from Proposition 6.1 that Γ is connected if $G = \langle J, K \rangle$. The long neighbors of x are the J-conjugates of y since

$$x^{\perp} = \{y^g\colon\ g \in C_G(x)\} = \{y^g\colon\ g \in J\}.$$

Likewise the short neighbors of x are the J-conjugates of x_1. In other words, the long (respectively short) neighbors of x are the long (respectively short) root reflections of $J \cong W(B_3)$. By assumption, $x \perp x_2$, $x_1 \perp x_2$ and likewise $y \perp y_2$, $y_1 \perp y_2$ in Γ. Since $(x, y)^{x_2 y_2} = (x, y_1)^{y_2} = (x_1, y_1)$, the vertices x_1 and y_1 are adjacent as well.

In particular, (x, x_1, x_2), (x, x_1, y), (x, y, y_1), (y, y_1, y_2) are ordered triangles of Γ. We claim that G acts transitively on oriented triangles of the same type. Indeed, let (a, b, c) be an oriented triangle of Γ where a, b, c are short vertices. Let $g \in G$ such that $(a, b) = (x, x_1)^g$ and set $d = {x_2}^g$. The vertices b, c, d are short neighbors of a, and hence short root reflections in J^g. By Example 6.4, we find $h \in J^g$ such that $(b, c) = (b, d)^h$. Therefore $(a, b, c) = (x, x_1, x_2)^{gh}$. Now, let (a, b, c) be an oriented triangle of Γ where a is a short and b, c are long vertices. Let $g \in G$ such that $(a, b) = (x, y)^g$. Set $d = {y_1}^g$. b, c, d are long neighbors of a, and therefore long root reflections of J^g. By assumption, b commutes with c as well with d. Consequently, $c = d$, see Example 6.4. Thus $(a, b, c) = (x, y, y_1)^g$. Analogously for triangles of the other types.

We observed that the long (respectively short) neighbors of x are the long (respectively short) root reflections of J. Two long neighbors a, b of x are adjacent if and only if (x, a, b) is a triangle in Γ. By transitivity on oriented triangles of the same type, this is the case if and only if we find $g \in G$ such that $(x, a, b) = (x, y, y_1)^g$, or, equivalently, if and only if we find $g \in J$ such that $(a, b) = (y, y_1)^g$. Two long root reflections $a, b \in J$ are conjugate to the two commuting long root reflections y, y_1 if and only if they commute themselves. Thus the long induced subgraph of $x^\perp$ is isomorphic to the long induced subgraph of $\mathbb{W}(B_3)$. Likewise, let a be a short and b a long neighbor of x. Again, a and b are adjacent if and only if (x, a, b) is a triangle which is equivalent to finding $g \in J$ such that $(a, b) = (x_1, y)^g$. By Example 6.4, this is the case if and only if a and b commute. Finally, let a, b be two short neighbors of x. By Example 6.4, we always find $g \in J$ such that $(a, b) = (x_1, x_2)^g$. Hence, a and b are adjacent. Since Γ was constructed to be transitive on short vertices this completes the proof that Γ has short local graph $\mathbb{W}(B_3)$. Likewise, we find that the long local graph of Γ is $\mathbb{W}(C_3)$. □

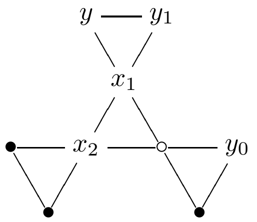
Since $\mathbb{W}(F_4)$ is not locally recognizable, compare Theorem 5.12, we need to add additional assumptions to those given in Proposition 6.5 to deduce that the graph Γ defined in Proposition 6.5 is not only locally like $\mathbb{W}(F_4)$ but actually is isomorphic to either $\mathbb{W}(F_4)$ or Γ_{24b}. In both cases we are then able to deduce that G is isomorphic to the Weyl group $W(F_4)$.

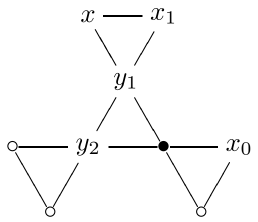
Theorem 6.6. *Let G be a group with nonconjugate involutions x, y such that*

- $C_G(x) = \langle x \rangle \times J$ *with* $J \cong W(B_3)$,
- $C_G(y) = \langle y \rangle \times K$ *with* $K \cong W(C_3)$,
- *x (respectively y) is a short (respectively long) root reflection in K (respectively J),*
- *$J \cap K$ contains involutions x_1, y_1 such that x_1 (respectively y_1) is a short (respectively long) root reflection in K as well as in J, and*
- *there are a long root reflection $y_0 \neq y, y_1$ in J and a short root reflection $x_0 \neq x, x_1$ in K such that x_0 and y_0 commute.*

If $G = \langle J, K \rangle$ then $G \cong W(F_4)$.

Proof. Observe that y_0 does not commute with x_1 and that x_0 does not commute with y_1. Set $x_2 = {x_1}^{y_0}$ as well as $y_2 = {y_1}^{x_0}$. Consequently, $x_2 \neq x_1$ is a short root reflection in J, and $y_2 \neq y_1$ is a long root reflection in K. Summarizing, the elements x_1, x_2 are short root reflections in $J \cong \mathbb{W}(B_3)$ and y, y_1, y_0 are long root reflections in J. Likewise, y_1, y_2 are long root reflections in $K \cong \mathbb{W}(C_3)$ and x, x_1, x_0 are short root reflections in J. Accordingly, the Weyl graphs of J and K are as follows.





Using Example 6.4, it follows that $x_1 = x^{y_2}$ and $y_1 = y^{y_2}$. Moreover, $(x_0\, x_1)^3 = 1$ or, equivalently, ${x_0}^{x_1} = {x_1}^{x_0}$, and likewise ${y_0}^{y_1} = {y_1}^{y_0}$. Using that x and y_0 commute, and that

$$(x, y_0)^{x_0\, y_1\, x_0\, y_0} = (x^{x_2\, y_0}, {y_1}^{y_0\, x_0\, y_0}) = ({x_1}^{y_0}, {y_1}^{x_0}) = (x_2, y_2),$$

we infer that x_2 and y_2 commute as well. Consequently, $y_2 = {y_2}^{x_2}$ is a long root reflection in both K and K^{x_2}. Analogously, x_2 is a short root reflection in both J and J^{y_2}. The assumptions of Proposition 6.5 are therefore fulfilled. Hence the graph Γ as defined in Proposition 6.5 is connected and locally like $\mathbb{W}(F_4)$.

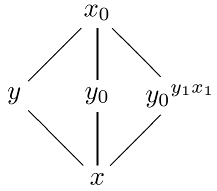
Recall that $x = {x_1}^{x_0 y_1 x_0}$, $y = {y_1}^{y_0 x_1 y_0}$, and ${x_0}^{x_1} = {x_1}^{x_0}$, ${y_0}^{y_1} = {y_1}^{y_0}$. Therefore,

$$(x_0, y_0)^{x_1 y_1 x_0 y_0} = ({x_1}^{x_0 y_1 x_0 y_0}, {y_1}^{y_0 x_1 y_0 x_0}) = (x^{y_0}, y^{x_0}) = (x, y)$$

which shows that x_0 and y_0 are adjacent in Γ. Notice, for instance using Example 6.4, that ${y_0}^{x_1 y_1}$ is a long root reflection of J not commuting with y and y_0. By assumption, x and y_0 are adjacent. Since

$$(x_0, {y_0}^{x_1 y_1})^{x_1 y_1 x_0 y_0} = (x, y_0),$$

the vertices x_0 and $y_0^{y_1 x_1}$ are adjacent as well. We therefore have the following induced subgraph.



Accordingly, $\mu(x, x_0) = 3$ where μ is as introduced in Proposition 5.10. Likewise, we find $\mu(y, y_0) = 3$. By construction, G acts transitively on vertices of the same type. Further, recall from Example 6.4 that $J \cong W(B_3)$ acts transitively on ordered pairs of nonadjacent vertices of the same type contained in $x^{\perp}$. Therefore G acts transitively on oriented 3-paths of the same type.

It follows from Proposition 5.20 that $\mu = 3$. Theorem 5.12 therefore applies, and Γ is isomorphic to $\mathbb{W}(F_4)$ or to Γ_{24b}. In either case, $\mathrm{Aut}(\Gamma)$ is isomorphic to $W(F_4)/Z$.

Note that J is generated by x_1, y_1, y_0, and likewise K is generated by x_1, y_1, x_0. The conjugates of x and y therefore generate $G = \langle J, K\rangle$ and Proposition 6.1 implies that $G/Z(G)$ acts faithfully on Γ. $Z(G) \leqslant C_G(x)$ and hence $Z(G) \leqslant Z(C_G(x)) = \langle x\rangle \times Z(J)$. Since $x \notin Z(G)$ we find that $|Z(G)| \leqslant |Z(J)| = 2$. While the stabilizer of a vertex in $\mathrm{Aut}(\Gamma)$ has order 48, the stabilizer in G has order 96. We therefore conclude that $|Z(G)| = 2$. Since $G/Z(G)$ acts transitively on Γ, the orbit-stabilizer formula implies that $G/Z(G)$ is isomorphic to $\mathrm{Aut}(\Gamma)$. Thus $G/Z(G) \cong W(F_4)/Z$. With $|G| = 1152$ in mind, the subsequent remark shows that in fact $G \cong W(F_4)$. □

Remark 6.7. There is a another approach, suggested to us by Hendrik Van Maldeghem, to proving Theorem 6.6 which does not rely on our previous graph-theoretical recognition results. Suppose that the assumptions of Theorem 6.6 are satisfied. By finding appropriate generators and relations for G we will prove that G is a quotient of $W(F_4)$. Recall from the proof of Theorem 6.6 that J is generated by x_1, y_1, y_0, and likewise K is generated by x_1, y_1, x_0. Accordingly,

$$J = \langle x_1, y_0, y_1 \colon\ x_1^2 = y_0^2 = y_1^2 = (x_1 y_0)^4 = (y_0 y_1)^3 = (y_1 x_1)^2\rangle,$$

and

$$K = \langle y_1, x_0, x_1 \colon\ y_1^2 = x_0^2 = x_1^2 = (y_1 x_0)^4 = (x_0 x_1)^3 = (x_1 y_1)^2\rangle.$$

Since $G = \langle J, K \rangle$ we have $G = \langle x_0, x_1, y_0, y_1 \rangle$, and all the above relations hold. Recall from the proof of Theorem 6.6 that $x = {x_1}^{x_0 \, y_1 \, x_0}$ and $y = y_1^{y_0 \, x_1 \, y_0}$. By assumption, y_0 commutes with x, and x_0 commutes with y. Further, we assumed that x_0 and y_0 commute. Summarizing, this accounts for the following additional relations.

$$\begin{aligned} ({x_1}^{x_0 \, y_1 \, x_0} \, y_0)^2 &= 1, \\ (x_0 \, y_1^{y_0 \, x_1 \, y_0})^2 &= 1, \\ (x_0 \, y_0)^2 &= 1. \end{aligned}$$

The free group generated by four elements x_0, x_1, y_0, y_1 together with these relations (actually we can omit one of the first two) and the relations stated above for J and K is isomorphic to $W(F_4)$. A proof of this statement using GAP, see [GAP07], can be found in the appendix of [Str08].

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